

## PID Loop Shaping

This Interactive Learning Module (ILM) is part of a sequence of modules designed to make you familiar with PID control. The other modules are *PID Tuning* and *PID Windup*. The modules were designed as a complement to the book [1]. They can be viewed as an attempt to make the key pictures in the book interactive. This particular module is designed to develop intuition for loop-shaping design of PID controllers. There are two icons to access *Instructions* and *Theory*. Instructions give access to this document which contains suggestions for exercises. Theory gives access to help via Internet.

Loop shaping is a design method where it is attempted to choose a controller such that the loop transfer function obtains the desired shape. In this module the loop transfer function is illustrated by its Nyquist plot. The module shows the Nyquist plots of the process transfer function  $P(s)$  and the loop transfer functions  $L(s) = P(s)C(s)$ . It can be shown how the loop transfer functions changes when controller parameters are changed and conversely what parameters are required to obtain a given shape of the loop transfer function.

The key idea is that the action of the controller can be interpreted as mapping the process Nyquist plot to the Nyquist-plot of the loop transfer function. For PI and PD control the mapping can be uniquely represented by mapping only one point. This point on the Nyquist plot of the process, which is called the *design point*, is characterized by its frequency. The corresponding point on the loop transfer function is called the *target point*. For PID control it is also possible to have an arbitrary slope of the loop transfer function at the target point. The module makes it possible to see the effects of the controller parameters on the map and also the effects of moving the target point.

The module can be used in many different ways, one extreme is a full-fledged exercise with serious analysis and reporting, another is simply free experimentation following your own ideas. Whatever you do it is good practice to record your observations. The modules can also be used in lectures. The following exercises are intended to give a good intuitive understanding of the properties of PID Control. The exercises are structured as follows:

1. Getting started
2. Effect of controller parameters
3. Effect of design and target points
4. Phase margin constraint
5. Gain margin constraint
6. Sensitivity constraint

### 1. Getting Started

Get familiar with the module by exploring the menu *Settings* and by manipulating the interactive elements. If you are reading this you are already familiar with the icon *Instructions*. A default process transfer function is loaded when you start the system. The Nyquist plots of the process (in black) and the loop transfer function (in red) appear to the right. Above the plot there are buttons *save* and *delete* at the top right-hand corner of the screen that makes it easy to store a simulation for comparison.

- Explore the menus, observe what happens when you change the design point of the Nyquist plot of the process and the target point of the Nyquist plot of the loop transfer function.
- Change the scales by moving the triangular mark on the vertical line.

Several options can be chosen from the *Settings* menu which has the following entries

- Process transfer function
- Parameters
- Load/Save
- Examples

The process transfer function can be chosen from the entry *Process transfer functions*. Notice that it is possible to choose transcendental functions and that you also can enter a transfer function in string format like 'exp(-sqrt(s))'. In this way we can introduce dynamics of processes described by partial differential equations. A process transfer function can be introduced by specifying poles and zeros graphically. Numerical values of the parameters can be given using the selection *Parameters* in the *Settings* menu. Results can be stored and recalled using the *Settings>Load/Save* menu. Data can be saved and recalled using the options *Save design* and *Load design* in the *Settings>Load/Save* menu. The option *Save report* in the same menu can be used to save all essential data in ascii format. This is useful for documenting your results. Figures from the book [1] can be obtained using the selection *Examples* in the *Settings* menu. In this way you can explore what happens when the parameters in the examples are changed.

To illustrate loop shaping you can choose the design point graphically by dragging it in the Nyquist plot. It can also be changed by the slider for *wdesign* where the point is specified by the corresponding frequency. The target point on the Nyquist plot of the loop transfer function and the slope at the target point can also be dragged and you can observe the controller parameters required to the particular shape of the loop transfer function. It is also possible to constrain the target point to the unit circle (Pm), the negative real axis (Gm), and the circles representing constant sensitivity (Mt), constant complementary sensitivity (Mt) or a circle that encloses the sensitivity circles (M).

Parameters which characterize robustness and performance are also presented numerically.

## 2. Effect of Controller Parameters

The purpose of this exercise is to show how controller parameters influence the loop transfer function. Controllers can be represented in many different ways. In this exercise we use the parameterization

$$C(s) = k + \frac{k_i}{s} + k_d s.$$

Let the process transfer function be  $P(s)$ . The loop transfer function is thus

$$L(s) = kP(s) + \left(\frac{k_i}{s} + k_d s\right)P(s).$$

The point on the Nyquist curve of the loop transfer function corresponding to the frequency  $\omega$  is thus given by

$$L(i\omega) = kP(i\omega) + i\left(-\frac{k_i}{\omega} + k_d\omega\right)P(i\omega). \quad (1)$$

The proportional gain thus changes  $L(i\omega)$  in the direction of  $P(i\omega)$ , integral gain  $k_i$  changes it in the direction of  $-iP(i\omega)$  and derivative gain  $k_d$  changes it in the direction of  $iP(i\omega)$ .

In the module the design point is marked by a circle on the process transfer function. The point can be changed by dragging or by using the slider marked (wdesign). The controller gains  $k$ ,  $k_i$ , and  $k_d$  can be adjusted directly by dragging arrows or from the sliders.

Some exercises are given below. Use the *Save report* feature in the menu *Settings>Load/Save* to save interesting results.

- Choose the transfer function

$$P(s) = \frac{1}{(s + 1)^4}$$

from the *Settings>Process Transfer Function* menu. Select a point on the process transfer function where the phase is  $-90^\circ$  as the design point. Choose proportional control by clicking on P, change the controller gain and observe how the loop transfer function changes. Repeat for different design points.

- Determine the value of the gain for a proportional controller where the closed-loop system changes from stable to unstable. Verify the value analytically. Repeat for integral control (I).
- Select a PI controller and explore how the loop transfer function changes with the controller parameters and the selected design point on the Nyquist plot of the process transfer function.
- Select a PID controller and explore how the loop transfer function changes with the controller parameters and the selected design point on the Nyquist plot of the process transfer function.
- Select a PI controller, fix a value of the integral gain, then adjust the proportional gain until you have obtained a controller with  $M_s < 1.4$ . Compare the results with Example 7.2 in [1].
- Choose the process transfer function

$$P(s) = \frac{1}{(s + 1)(s^2 + 0.06s + 0.5)}$$

and the PID controller

$$C(s) = 0.038 + \frac{0.209}{s} + 0.38s.$$

Evaluate gain and phase margins and the maximum sensitivities. This is an example of a system which has poor robustness in spite of the fact that the gain and phase margins are quite reasonable.

### 3. Effect of Design and Target Points

This exercise will give you a feel for the controller parameters required to obtain a given shape of the loop transfer function. In this and the following experiments you will choose the design and target points. This is accomplished by clicking on the box Pm (phase margin) on the screen. You can then select the target point but it is restricted to lie on the unit circle. The controller parameters required are then computed. The computations are described in the Appendix.

Here are some exercises:

- Choose the transfer function

$$P(s) = \frac{1}{(s + 1)^4}$$

from the *Settings>Process Transfer Function* menu. Select a point on the process transfer function where the phase is  $-90^\circ$  as the design point. Move the target point around and observe the controller parameters and the shape of the loop transfer function.

- Move the target point close to  $-0.5 - 0.5i$  and observe what happens when you change the slope of the loop transfer function at the target point by dragging or by using the slider for slope. Look for the values where controller parameters change sign. What happens when integral gain  $k_i$  becomes negative?

### 4. Phase Margin Constraint

In this and the following exercises you will move the target point but the choice of target points will be constrained. In this particular case the target point is restricted to lie on the unit circle which means that the target point will directly specify the phase margin.

- Choose the transfer function

$$P(s) = \frac{1}{(s + 1)^4}$$

from the *Settings>Process Transfer Function* menu. Select a point on the process transfer function where the phase is  $-90^\circ$ . Explore the shape of the loop transfer function when different points and slopes are chosen.

- What happens if you choose a very small  $w_{design}$ ? Hint: look at the shape of the loop transfer function and the controller parameters. Can you explain analytically what happens?
- What does it mean when the loops appear on the loop transfer function?
- Select a phase margin of  $60^\circ$ , change the design point on the process Nyquist plot and the slope to maximize integral gain subject to the constraint that  $M_s \leq 1.6$ .

## 5. Gain Margin Constraint

In this exercise the target point on the loop transfer function is restricted to move on the real axis. This is accomplished by clicking on the box marked Gm gain margin which constrains the target point to be on the real axis.

- Choose the transfer function

$$P(s) = \frac{1}{(s + 1)^4}$$

from the *Settings>Process Transfer Function* menu. Choose a gain margin of 3, and adjust the design point on the process Nyquist plot and the slope to maximize integral gain, subject to the constraint  $M_s \leq 2$ .

## 6. Sensitivity Constraints

In this exercise the target point on the loop transfer function is restricted to move on a sensitivity circle. Three choices are available, the sensitivity circle ( $M_s$ ), the complementary sensitivity circle ( $M_t$ ) and the smallest circle that encloses both sensitivity circles ( $M$ ). These circles are defined as follows:

Contour	Center	Radius
$M_s$ -circle	-1	$1/M_s$
$M_t$ -circle	$-\frac{M_t^2}{M_t^2 - 1}$	$\frac{M_t}{M_t^2 - 1}$
$M$ -circle	$-C_{st}$	$R_{st}$

where

$$x_1 = \max\left(\frac{M_s + 1}{M_s}, \frac{M_t}{M_t - 1}\right), \quad x_2 = \max\left(\frac{M_s - 1}{M_s}, \frac{M_t}{M_t + 1}\right)$$

$$C_{st} = \frac{x_1 + x_2}{2}, \quad R_{st} = \frac{x_1 - x_2}{2}$$

You select the circle by clicking on the boxes marked  $M_s$  for the sensitivity circle,  $M_t$  for the complementary sensitivity circle and  $M$  for the combined circle. Here are some exercises:

- Choose the transfer function

$$P(s) = \frac{1}{(s + 1)^4}$$

from the *Settings>Process Transfer Function* menu. Choose the constraints  $M_s \leq 1.6$  and  $M_t \leq 1.2$ . Adjust the target point on the combined sensitivity circle  $M$ , the design frequency, and the slope so that the integral gain is maximized.

## 7. Examples

There are two examples from the book [1] that can be loaded. These examples illustrate some extreme cases of PI control and loop shaping.

- Load Figure 6.24 from the *Settings>Examples* menu. The system has the transfer function

$$P(s) = \frac{9}{(s+1)(s^2+as+9)}$$

with  $a = 0.5$  and the controller is a PI controller. Explore the sensitivity of the system for small changes in the controller parameters. Also investigate the case  $a = 0.2$ .

- Load Figure 6.25 from the *Settings>Examples* menu. The system has the transfer function

$$P(s) = \frac{1}{(s+1)^4}$$

and the controller is a PID controller. Find the maximum sensitivities, the gain and phase margins. Explore the sensitivity of the system to small changes in the controller parameters.

## 8. Reference

- [1] Karl Johan Åström and Tore Hägglund. *Advanced PID Control*. ISA - The Instrumentation, Systems, and Automation Society, Research Triangle Park, NC 27709, 2005.

## Appendix

Let the design point be characterized by the frequency  $\omega$  and let the target point be  $L_x + iL_y$ . Introduce  $\rho = |P(i\omega)|$ ,  $\varphi = \arg P(i\omega)$  and solve (1) for  $k$  and  $k_i$  we get

$$\begin{aligned}k &= \frac{L_x \cos \varphi + L_y \sin \varphi}{\rho} \\k_i - \omega^2 k_d &= \frac{\omega(L_x \sin \varphi - L_y \cos \varphi)}{\rho} \\k_d - \frac{k_i}{\omega^2} &= \frac{-L_x \sin \varphi + L_y \cos \varphi}{\rho \omega} = A\end{aligned}$$

For PI control we have  $k_d = 0$  and the parameters  $k$  and  $k_i$  are then uniquely determined. For PID control the proportional gain is uniquely given but it is necessary to have another condition to determine integral and derivative gains. This condition can be obtained by specifying the angle  $\alpha$  of the loop transfer function at the target point. Straight forward but tedious calculations give

$$k_d + \frac{k_i}{\omega^2} = \frac{(L_x \rho' / \rho - L_y \varphi') \sin \alpha - (L_x \varphi' + L_y \rho' / \rho) \cos \alpha}{a \cos \alpha + b \sin \alpha} = B$$

where

$$P = \rho(\omega)e^{i\varphi(\omega)} = a(\omega) + ib(\omega)$$

The slope of the loop transfer function is given by

$$L'(i\omega) = P'(i\omega)C(i\omega) + P(i\omega)C'(i\omega)$$

where

$$C'(i\omega) = i\left(k_d + \frac{k_i}{\omega^2}\right)$$

Since  $PC = L = L_x + iL_y$  we have

$$L' = \frac{P'PC}{P} + PC'$$

Introducing  $P = \rho e^{i\varphi} = a + ib$  we find

$$\frac{P'}{P} = (\log P)' = \frac{\rho'}{\rho} + i\varphi''$$

and we get

$$L' = L_x \frac{\rho'}{\rho} - L_y \varphi' - b\left(k_d + \frac{k_i}{\omega^2}\right) + i\left(L_x \varphi' + L_y \frac{\rho'}{\rho} + a\left(k_d + \frac{k_i}{\omega^2}\right)\right)$$

If the slope is  $\alpha$  we get

$$\frac{L_x \varphi' + L_y \rho' / \rho + a\left(k_d + \frac{k_i}{\omega^2}\right)}{L_x \rho' / \rho - L_y \varphi' - b\left(k_d + \frac{k_i}{\omega^2}\right)} = \frac{\sin \alpha}{\cos \alpha}$$

Hence

$$k_d + \frac{k_i}{\omega^2} = \frac{(L_x \rho' / \rho - L_y \varphi') \sin \alpha - (L_x \varphi' + L_y \rho' / \rho) \cos \alpha}{a \cos \alpha + b \sin \alpha}$$

The controller gains are then given by

$$\begin{aligned} k &= \frac{L_x \cos \varphi + L_y \sin \varphi}{\rho} \\ k_i &= \frac{-A + B}{2} \omega^2 \\ k_d &= \frac{A + B}{2} \end{aligned} \tag{2}$$

This equation is used to compute the controller gains that give the desired target point with a given slope of the loop transfer functions.